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IMPULSIVELY LOADED STRAIN HARDENED RATE-SENSITIVE RINGS AND TUBES

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Abstract—The response of rate-sensitive perfectly plastic and linear strain hardened rings and tubes are calculated over a wide range of parameters. Very accurate numerical solutions are obtained and these compare very favorably with approximate solutions based on constant initial strain rate properties. The results provide conclusive support for the approximate techniques. In addition, the response curves are presented in a form convenient for direct engineering use.

INTRODUCTION

IN MANY important engineering problems rate sensitivity in dynamic structural response presents a first order effect which is, unfortunately, analytically elusive. To cite specific examples: containment vessels for nuclear reactors must be able to sustain certain explosive impulses; explosive metal forming technology has a manifold of design problems; design of meteoroid bumpers to sustain hypervelocity impact is yet another; and the general dynamic response of structures to blast loading situations is a final all-inclusive example. The difficulty of including rate sensitivity aspects of material behavior in a rational way in the many important practical design problems is associated with the very strong nonlinearity present in the constitutive relations. Rigorous analyses of relatively simple structural elements (beams) have been made and a sampling of these shows clearly the extreme difficulties present [1, 2]. (An excellent survey of beam studies has been carried out [3]).

Recent findings suggest that simplified engineering analyses may now be made for certain classes of problems. For perfectly plastic rate-sensitive rings an approximation based on the fundamental physics of the situation was introduced which was capable of generalization to more complex problems [4]. It consists of assuming that the impulsively loaded ring has a yield stress which is a constant associated with its initial strain rate throughout the entire flow regime of the ring. The reason for the success of this approach is that the kinetic energy is converted into plastic work before the stress strain rate point moves appreciably from its initial position. Appropriate extension of this concept to cover strain hardening situations has also been considered [5]. For a pulse loaded rather than an impulsively loaded ring similar simplifications are possible and have essentially been achieved [6].

Ring tests have been performed at Arthur D. Little and Battelle Memorial Institute, the findings of which are consistent with the theoretical simplifications discussed [7, 8]. The simplified approach of [4] has been applied by Jones [9] to the large deflection response

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of rate-sensitive strain hardened beams, by Perrone [10] to impulsively loaded rate-sensitive plates (small deflection) and further by Jones [11] for impulsively loaded rate-sensitive strain hardened plates undergoing large deformations.

Response of rings and cylinders made of an elastic-strain hardening rate-sensitive material have been calculated [12], but a linear rate-sensitive model was utilized. Very capable computer programs using finite difference techniques in space and time have been devised for a number of structural elements with strain hardening, rate-sensitivity and large deformations [13–16].

Hitherto, in the work done to date only limited checks have been made between the results of exact solutions and simplified approximate procedures. In the present paper comparisons will be made between exact and approximate approaches for a wide range of parameters for strain hardening and perfectly plastic rate sensitive rings and tubes. In addition to giving comprehensive final verification of the simplifications discussed, the results should also be of significant engineering use for the problems considered.

In the next section the strain hardening rate sensitive flow laws are reviewed; in the third section the class of problems to be discussed are stated. The fourth section contains a discussion of how the exact solution is obtained. The next section demonstrates how the approximate solution is obtained. In the final two sections comparisons and discussions between the two approaches are given, and conclusions are drawn.

STRAIN HARDENING, RATE-SENSITIVE FLOW LAWS

The mathematical model demonstrating rigid linear strain hardening material behavior is shown in Fig. 1. The yield stress σ_0 and the strain hardening rate c are the two significant material parameters of interest. Should plastic strains be much larger than elastic strains a rigid plastic model should prove satisfactory. For a perfectly plastic material the strain hardening rate c would vanish so that this case can be considered as a special limiting case of the linear strain hardening material.

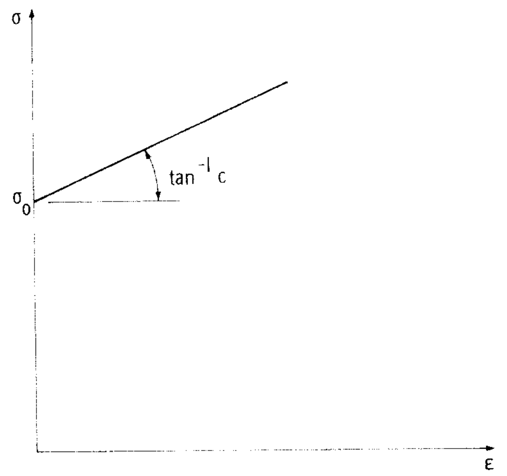

FIG. 1. Rigid linear strain hardening material.

A rate sensitive material is one for which the physical material parameters vary with the rate of deformation or strain rate. A special case of rate-sensitive behavior which has received considerable attention in the literature is one for which the yield stress is rate sensitive but the strain hardening rate is not. Whereas the variation of the yield stress σ_0 with strain rate is very prominent and has been determined for many structural materials, the variation of the strain hardening factor c with strain rate has the benefit of much less correlation with experimental results.

For perfectly plastic rate sensitive materials Cowper and Symonds [17] recommended the following flow law

$$\frac{\sigma}{\sigma_0} = 1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{1/n} \tag{1}$$

where

n and D are material constants

$\dot{\varepsilon}$ = strain rate

σ_0 = static yield stress

σ = dynamic yield stress.

This flow law does fit experimental results fairly well for a number of structural materials. This mathematical model has been utilized by many subsequent investigators who were concerned with the calculation of the response of various perfectly plastic rate-sensitive structural elements (all analytical papers listed under references).

A model of strain hardening rate sensitive material behavior recommended by Malvern [18] is shown in equation (2)

$$\dot{\varepsilon} = f[\sigma(\varepsilon) - \sigma_0] \tag{2}$$

wherein the strain rate is assumed to be a function of the so-called "dynamic overstress".

Both the Cowper–Symonds Law of equation (1) for perfectly plastic rate-sensitive materials and the Malvern Law of equation (2) for strain hardened rate-sensitive materials suffer in analytical applications from very strong non-linearities. This fault, of course, is not with the posed formulation of the mathematical law but is due primarily to the fact that the physical material parameters vary slowly with strain rate, that is in such a way that the variation can only be displayed on a logarithmic plot. Hence, any mathematical formulation of how the material behaves must reflect this fact and result in a significant non-linearity arising.

A product type strain hardening rate sensitivity law has been introduced [5, 19] which uncouples, so to speak, the hardening and rate sensitivity aspects. An example of this type of flow law is shown in equation (3a).

$$\frac{\sigma}{\sigma_0} = \left[1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{1/n}\right][1 + c\varepsilon] \tag{3a}$$

$$c = f(\dot{\varepsilon}). \tag{3b}$$

This form differs from the Malvern formulation of equation (2) in that the hardening and rate sensitivity functions are clearly delineated rather than being interwoven into a single formulation. This product type of formulation does not prohibit the possibility of a strain hardening function which varies with strain rate. Indeed, if such were the case, a second collateral relation should be addended, equation (3b).

A display of the product flow law is shown in Figs. 2 and 3 for two different sets of parameters. The different strain level curves running from 0 to 10 per cent are associated with the strain hardening aspect of flow. During any given flow condition the material would usually start at some point on the 0 per cent curve and jump (or rather vary continuously) to each higher strain hardening level curve for the appropriate current strain rate. The material which is rate sensitive but exhibits no strain hardening would lie along the 0 per cent curves.

STATEMENT OF PROBLEM

The physical structural element considered is that of the thin ring or tube shown in Fig. 4. Since rate sensitive hoop response is the dominant flow mechanism it makes little difference from an engineering viewpoint whether the element considered is a ring or a tube. In the case of the ring, flow occurs in the circumferential direction positively and in the axial and radial directions negatively. For the infinitely long tube, however, no axial flow occurs.

If we assume that the material is isotropically rate sensitive and of the Tresca type (such as in [10]) then a schematic of the yield curve is as displayed in Fig. 5. σ_r, σ_θ and σ_z are the principle directions in the radial, circumferential and axial directions, respectively.

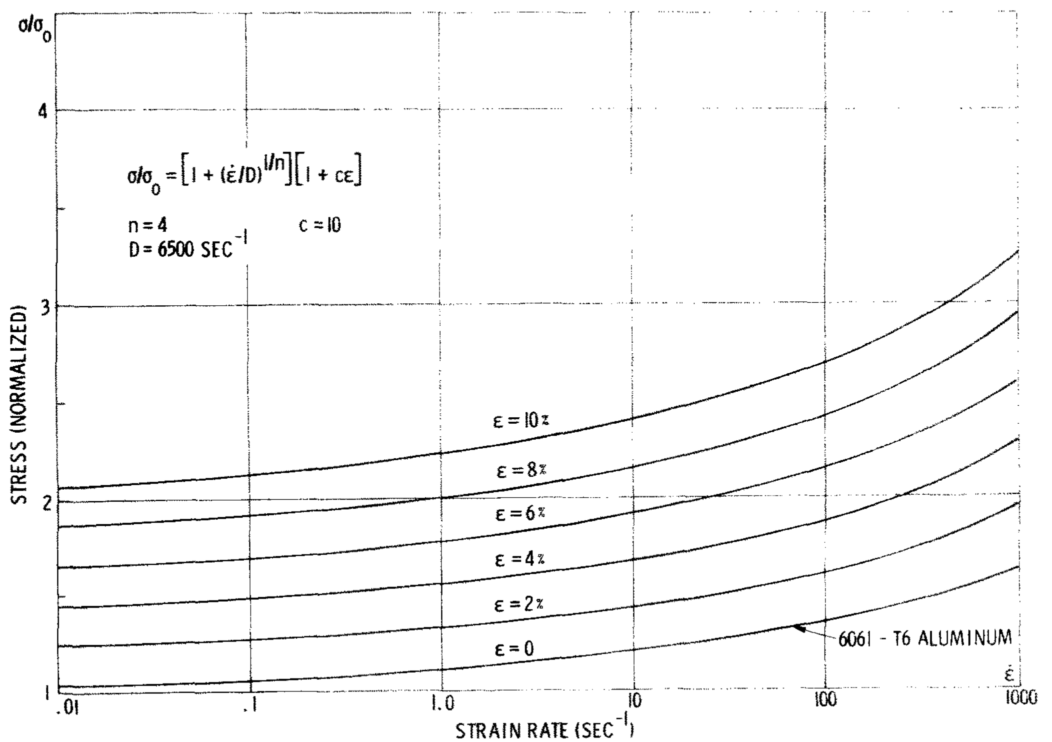

FIG. 2. Rate sensitive strain hardening with product law.

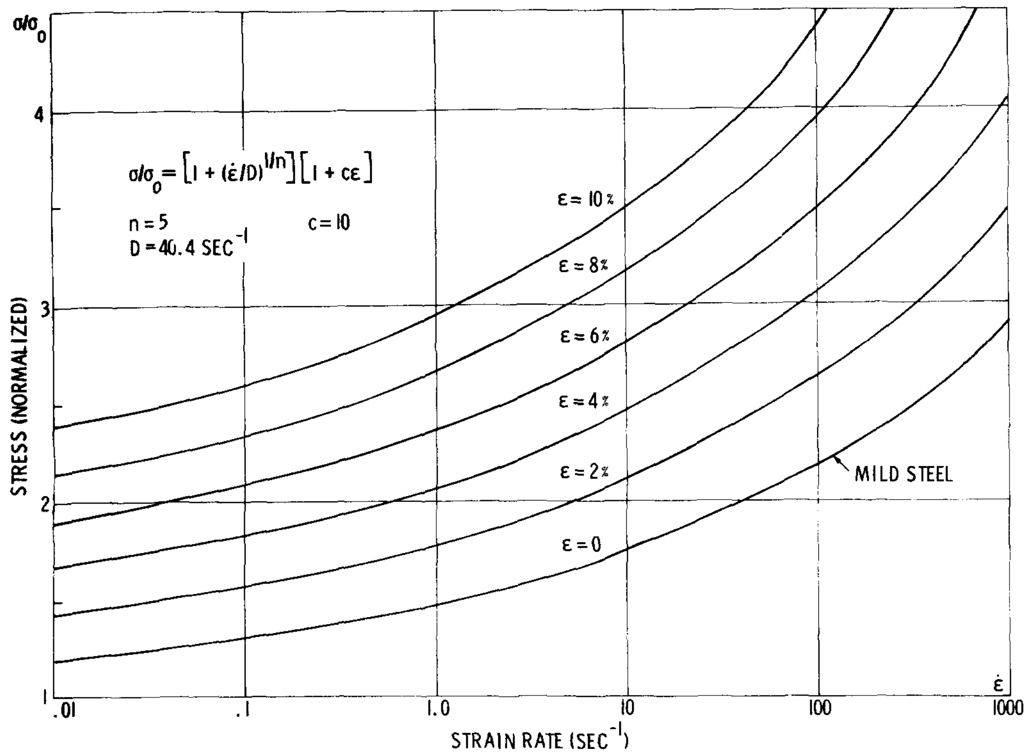

FIG. 3. Rate sensitive strain hardening with product law.

The dominant flow is, of course, in the circumferential direction. For the ring, flow occurs from face *BC* of the yield surface and the associated strain rate vector which is normal to the yield surface must point downward or below the $\sigma_z = 0$ plane (Fig. 5). For the case of infinitely long tubes, flow occurs from the face CD and the strain rate vector is

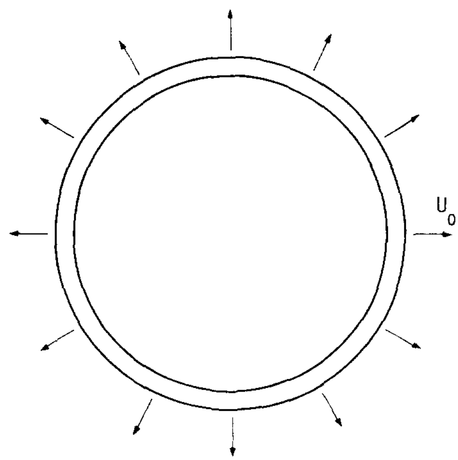

FIG. 4. Impulsively loaded ring or tube.

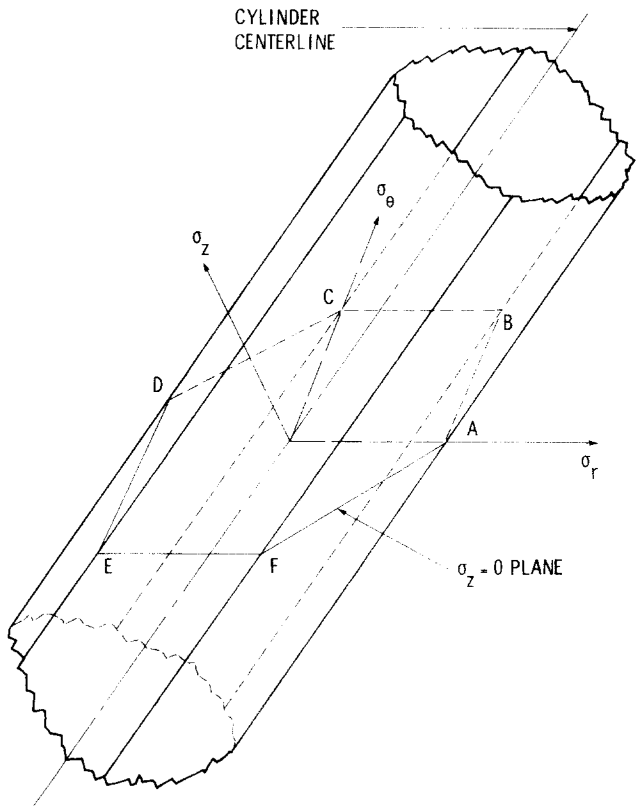

FIG. 5. Perspective of Tresca yield cylinder.

normal to this face making it parallel to the $\sigma_z = 0$ plane and above it (σ_z is positive). Because of rate sensitivity and strain hardening effects the yield surface could grow isotropically.

For thin rings and tubes the stress point lies approximately in the $\sigma_r = 0$ plane. The current amplitude of the yield surface, measured along the σ_θ axis, is dictated by the magnitude of $\dot{\varepsilon}_\theta$ and ε_θ which are related to the stress level via equations (3). As a consequence of this assumption, both rings and tubes are encompassed by the same formulation (as far as radial response is concerned).

The ring or tube, as the case may be, is impulsively loaded as shown in Fig. 4 with a uniform velocity U_0. The effect of the initial velocity U_0 is such to impart a kinetic energy to the structural element which is gradually dissipated into plastic work associated with circumferential rate sensitive stretching of the ring or tube. When the radial velocity and hence, kinetic energy reaches zero, motion ceases.

The material is assumed to obey a product strain hardening rate sensitive flow law as given by equations (3). The differential equation governing motion is readily derived [20] and the result is shown in equation (4).

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{2\sigma_0}{\rho U_0^2}\left[1 + \left(\frac{\dot{\varepsilon}}{D}\right)^{1/n}\right]\left[\frac{1 + c(r-1)}{r}\right] \tag{4}$$

where

$v = U^2/U_0^2$
U_0 = initial ring (tube) velocity
U = current ring (tube) velocity
r = current initial ring (tube) median radius
ρ = mass density.

In an equation of motion the independent variable would normally be time, but in equation (4) a minor transformation has been made to replace the original independent time variable by the dimensionless displacement variable r. In addition, for convenience the dependent variable has been replaced by v which is related to the square of the dimensionless velocity. For the impulsively loaded ring or tube the pertinent initial and final rest conditions are enumerated as follows:

Initial	Velocity = U_0	$v = 1$	$r = 1$
Final	Velocity = 0	$v = 0$	$r = r_f$

In equation (4) strains are always referred to the current deformation state. With this restriction it should be noted that fairly large deformations are possible. Within the limitations of the rigid plastic model utilized, the results are also valid for small deformations.

ACCURATE NUMERICAL (EXACT) SOLUTION

In this section a very careful numerical solution will be obtained to the differential equation of motion, equation (4). For the sake of distinguishing this solution from an approximate one to be obtained subsequently, this solution will be referred to as the "exact" solution.

For convenience, equation (4) is rewritten in the more suitable form of equation (5) as follows:

$$\mathrm{d}r = \frac{-\mathrm{d}v}{\left(\alpha + \dfrac{v^{1/2n}}{r^{1/n}}\gamma\right)\left(\dfrac{1+c(r-1)}{r}\right)} \tag{5}$$

where

$$\alpha = \rho U_0^2/(2\sigma_0)$$

$$\beta = U_0/(r_0 D)$$

$$\gamma = \alpha\beta^{1/n}.$$

A numerical solution to equation (5) by computer is readily obtained by replacing the differential quantities by finite changes and setting up a computer program to essentially accomplish the integration. As stated previously the initial values of r and v are unity. It should be recalled that v is related to the velocity squared and hence the kinetic energy. If we allow v to incrementally (or more precisely, decrementally) change by finite but very small steps, the associated changes in r can be easily calculated from the finite form of equation (5). Steps of 0·002 or 0·2 per cent are taken in v in uniform steps until v is finally equal to 0 corresponding to no motion. Each change in r has an effect on the denominator of equation (5) and the new value of r is used in the immediate subsequent step. As a result, the integration should be a fairly accurate one.

The same integration scheme could be readily utilized should the strain hardening constant be a function of strain rate, or indeed even if the strain hardening were non-linear.

Numerical results for an extensive number of cases are given in Figs. 6–9. Pertinent material parameters applying for each figure are displayed prominently. The curves correspond to the exact solutions obtained by computer. The ordinate in each case is the dimensionless final ring or tube deformation, and the abscissa is related to the initial kinetic energy. In each figure a series of curves are shown which apply to the material properties for the various initial strain rate conditions. The uppermost curve corresponds to the response of a ring or tube made of a rate insensitive material ($\sigma/\sigma_0 = 1$).

A quick perusal of these curves suggests a physical inconsistency, but a more penetrating look clarifies this potential paradox. Take for example Fig. 6; if one were to go along a fixed abscissa corresponding to a given initial kinetic energy, one observes that a lower final

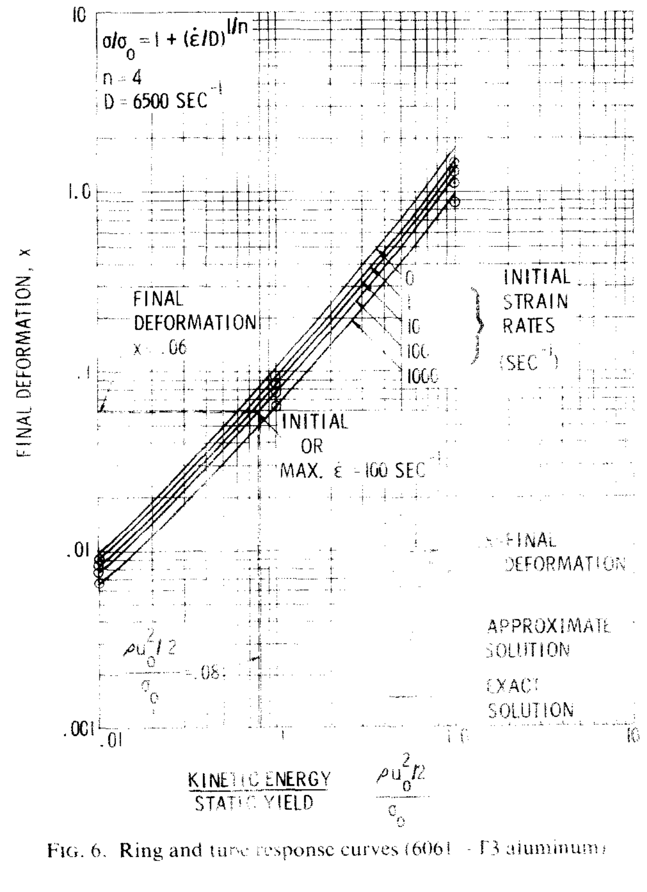

FIG. 6. Ring and tube response curves (6061 – T3 aluminum)

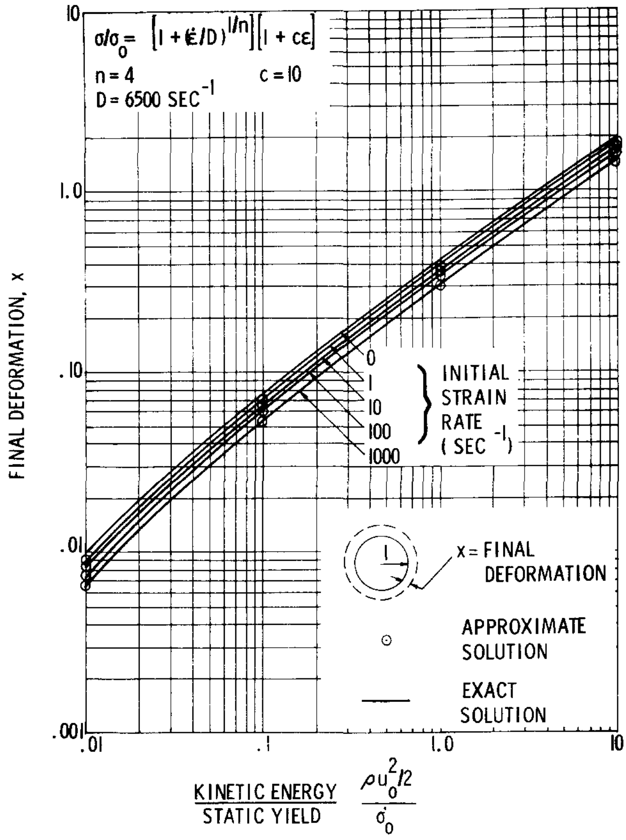

FIG. 7. Ring and tube response curves.

deformation would occur if the initial strain rate were 1000/sec rather than 1/sec. If one holds the material parameters fixed (for each figure this is the case) then the higher the initial strain rate the stronger the ring material, and hence the lower the final deformation.

To obtain a single point on any one of these curves would require a computer run as discussed earlier. Approximately 70 computer runs were required to obtain enough points to plot the four figures, Figs. 6–9.

We could make use of the results displayed from an engineering viewpoint in the following manner (see Fig. 6): for any given ring or tube calculate the initial kinetic energy and divide by the static yield stress. This specifies the value of the abscissa for any of the figures; go up vertically and examine where the initial kinetic energy value intercepts the initial strain rate level and read across to determine the final deformation. Some interpolation would be possible between the various figures for strain hardening values.

Should a ring or tube be pulse loaded as opposed to impulsively loaded, then, as discussed in [6] it would be possible to estimate the initial kinetic energy and strain rate by assuming the pulse load is impulsively applied. (The pressure level should be some multiple of the static collapse pressure). This observation in conjunction with Figs. 6–9 would give a technique for estimating the final deformation of pulse loaded rings or tubes.

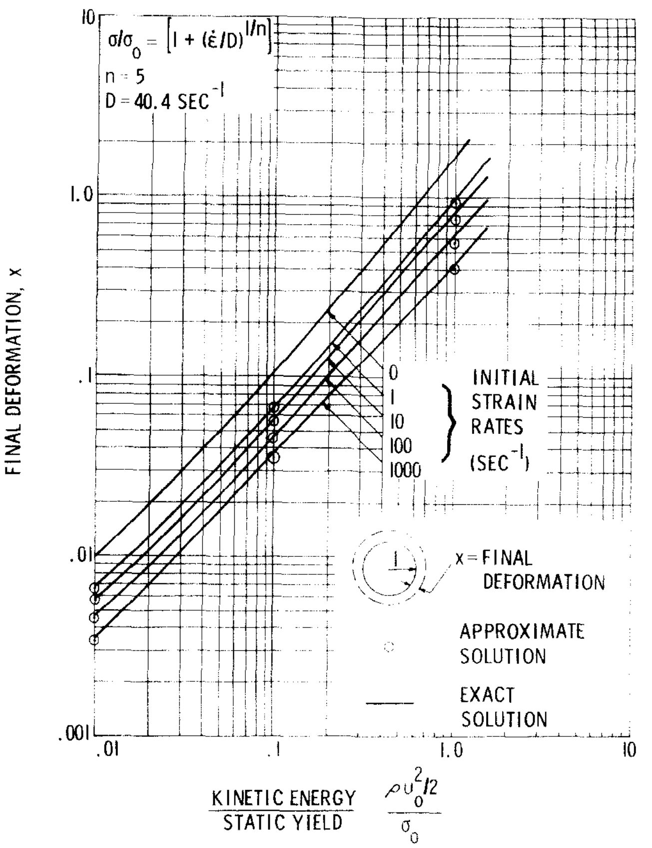

FIG. 8. Ring and tube response curves (mild steel).

APPROXIMATE SOLUTIONS

In the present section approximate solutions are obtained for perfectly plastic and strain hardened rate sensitive rings and tubes following the procedures discussed in [4] and [5]. To keep our discussion reasonably self contained we shall review the essence of these procedures.

In [4] it was observed by the author that for impulsively loaded rate-sensitive rings made of perfectly plastic materials the final deformation could be calculated by assuming the initial stress was a constant associated with the initial strain rate of the material. Comparisons in a few examples with a more exact solution based on a power series approach were very favorable, comparing within a few per cent. The success of this approximation is based on the observation that the ring kinetic energy is converted into plastic work before the stress strain rate point moves appreciably from its initial position.

It was recommended in [5] that a similar approximation could be utilized for strain hardening rate sensitive materials by assuming that flow occurs at an essentially constant strain rate. For example in Figs. 2 and 3 for the strain hardening situation, flow would occur along a vertical line. To date no comparisons have been made between this approximate procedure and very accurate results for strain hardened rate sensitive materials.

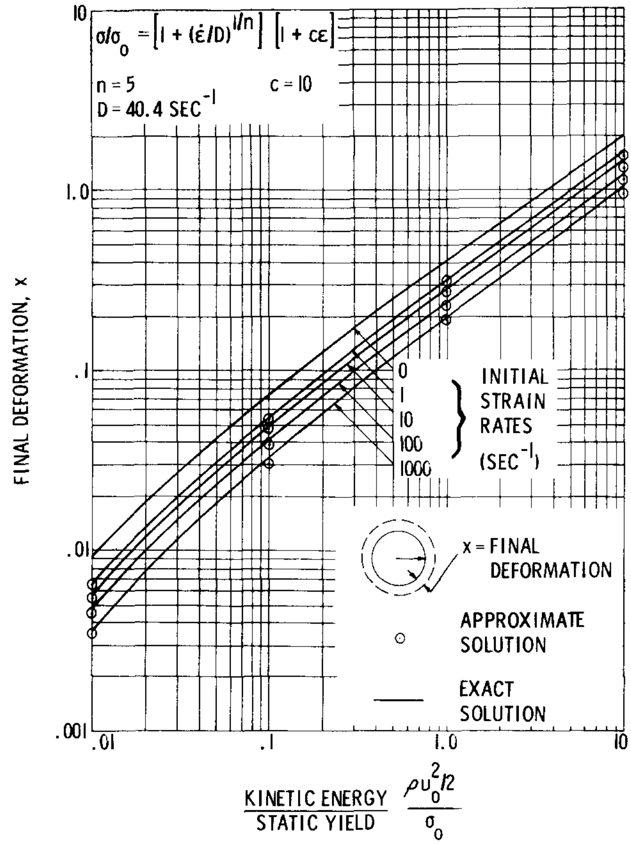

FIG. 9. Ring and tube response curves.

However, in [20] when discussing the ring test as a possible mechanism for determining rate-sensitive properties of materials, the author followed the recommendations made in [5] to calculate the approximate response of linear strain hardened rings.

We may make use of both of the above mentioned approximations by simply taking equation (5) and integrating the same with v in the denominator set equal to 1. Recall that v is the square of the velocity and its initial value is unity; therefore when we set v equal to 1 we are specifying that it take on its initial value. The term in which it appears is essentially the rate sensitive term. Should the material be perfectly plastic then c is equal to 0; if not c takes on its specified value.

With this qualification on v equation (5) may be rewritten as shown in equation (6).

$$\int_1^{r_f}\left(\alpha+\frac{\gamma}{r^{1/n}}\right)\left(\frac{1+c(r-1)}{r}\right)\mathrm{d}r=\int_1^0 -\mathrm{d}v. \tag{6}$$

Equation (6) is shown in a form ready for integration with the limits on the integration. The velocity is integrated between its initial unit value and 0, and the dimensionless radial displacement r is integrated between its initial value unity and its final value r_f. We could determine the final value of r (that is, r_f) by solving the integrated form of equation (6) for the situation where v is equal to 0. In other terms, solve for the value of r for which the integral on the left hand side of equation (6) vanishes.

For the various values of material parameters displayed in Figs. 6–9 a calculation of r_f was made and the results are shown throughout these figures by small circled dots. The comparisons between the exact and approximate solutions is obviously quite good for a wide range of parameters. A more extensive discussion of these comparisons is contained in the next section.

COMPARISONS BETWEEN EXACT AND APPROXIMATE SOLUTIONS

In Figs. 6–9 the final deformations are plotted vs. the initial kinetic energy for rings or tubes covering a wide range of initial strain rates and other material parameters. The smooth curves correspond to the exact numerical solution whereas the circled points refer to the approximate responses calculated by a much simpler technique.

The exact and approximate solutions differed by about 5 per cent. On the average the approximate solutions were of the order of 5 per cent smaller than the exact solution. With the approximation utilized it should be clear why the results should always be smaller. Since the material properties are assumed constant at their initial strain rate level, it should be clear that the approximate solution makes use of a stronger material than the exact one, and hence the final deformations should be smaller, as indeed they are.

We could attempt to bring both solutions into line with one another by adopting a uniform correction factor. Specifically if we increase all the approximate solutions by 5 per cent then the maximum deviations between the two solutions, that is between the approximate and the exact solutions, would always be less than 2 per cent.

We can readily assess the relative significance of strain hardening and rate sensitivity if we were to superpose the results of Figs. 6 on 7 or Figs. 8 on 9. By doing so and examining the final deformations for the two extremes, one with a strain hardening constant 0, the other for a strain hardening constant 10 it becomes apparent as to which are the more important zones for each effect. From these superpositions it is clear that for small deformations rate sensitivity effects are more important, whereas for larger deformations strain hardening effects become very significant. These same findings are evident in Jones' results of [9] in which he is concerned with the response of strain hardening rate-sensitive impulsively loaded beams.

CONCLUSIONS

Very accurate as well as approximate solutions are obtained for impulsively loaded perfectly plastic and strain hardened rate sensitive rings or tubes. These results are displayed in graphical form covering a wide range of parameters.

The very fine comparisons between the exact and the approximate solutions suggest that the approximate procedures can be used with confidence for strain hardening as well as for perfectly plastic materials. In point of fact, in the present paper the recommendation made in [5] to include strain hardening materials is here first assessed quantitatively and the results have been very definitely positive. The approximation suggested previously for perfectly plastic rate-sensitive materials and discussed in [4] may now be used with much greater confidence. Previous comparisons were only carried out for a few cases; in the present work a wide variety of comparisons have been successfully accomplished.

If we applied a uniform correction factor consisting of a 5 per cent increment of the approximate solutions then the resulting comparisons between the exact and the approximate solutions would always be less than 2 per cent over the entire range of parameters. When attempting to determine the approximate response of other structural elements it is likely that a similar uniform correction factor could be applied.

With respect to the relative significance of strain hardening and rate sensitivity effects on the final ring or tube deformations, the present findings are similar to those of Jones [9]. For small deformations rate sensitivity effects are more important than strain hardening and vice versa for large deformations.

The results of the present analysis for strain hardened material response suggests that prior work [6] on pulse loaded perfectly plastic rate sensitive ring response could be readily extended to account for strain hardening materials. For pressure pulse loaded rings or tubes one would have to estimate the strain rate level by assuming the pulse is applied impulsively. The appropriate equations of motion can then be readily integrated in the loading and unloading phase by accounting exclusively for strain hardening effects (at a constant strain rate).

While discussing further possible extensions of the present effort it should be mentioned that non-linear strain hardening could obviously be accounted for directly with the procedures discussed in the present paper. The integration of the associated equations would be a little more difficult, but not seriously so. It should also be noted that c, the strain hardening factor conceivably could be a function of strain rate. In the exact computer approach this variation could be taken care of directly; in the approximate approach one would accept the value of c associated with the initial strain rate and take this to be a constant.

The usual restrictions should be noted on the use of a rigid plastic vs. an elastic plastic model of material behavior. Plastic strains should be much larger than elastic strains.

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Абстракт—Определяется, для широкого круга параметров, поведение чуствительных и скорости идеально пластических колец и труб с линейным упрочнением. Получаются очень точные численные решения, которые сравниваются очень удобно с приближенными решениями, основанными на свойствах постоянной начальной скорости деформации. Результаты дают убедительное подтверждение для приближенных способов расчета. Затем, даются кривые поведения в удобной форме, для непосредственного использования в инжинерной практике.